JOURNAL OF APPROXIMATION THEORY 73, 343–356 (1993)

Best Approximation with Linear Constraints: A Model Example

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Communicated by Zeev Ditzian

Received June 26, 1991; accepted in revised form February 28, 1992

We consider best approximation in $L_p(\mathbb{R})$, $1 \leqslant p \leqslant \infty$, by means of entire functions y of exponential type subject to additional constraints $\Gamma_j(y) = 0$, $j = 1, ..., K$. Here Γ_j are (unbounded) linear functionals of the form $\Gamma_j(y) = D^n y(s_j) - \sum a_k D^k y(s_j)$ where s_j are fixed points. © 1993 Academic Press, Inc.

1. INTRODUCTION

The concept of "best approximation" is one of the most important in approximation theory. As examples we mention best approximation by algebraic or trigonometric polynomials. In general one considers quasi-normed spaces X and $\mathscr{E} \subset X$ and the best approximation functional

$$E(\lambda, x) = E(\lambda, x; X, \mathscr{E}) = \inf\{ \|x - y\|_X : \|y\|_{\mathscr{E}} < \lambda \}.$$

Here $\| \ \|_X$ and $\| \ \|_{\mathscr{E}}$ denote the quasi-norms of X and $\mathscr{E}$, respectively.

(*Remark.* We use the word quasi-norm in the sense defined in [1, Sect. 3.10]. Thus we only require positivity, $\|-x\| = \|x\|$, and the quasi-triangle inequality, i.e., $\|x + y\| \leqslant c(\|x\| + \|y\|)$ where $c \geqslant 1$.)

One of the main problems is to characterize the space $E_\alpha = E_\alpha(X, \mathscr{E})$ of all $x \in X$ such that $E(\lambda, x) = O(\lambda^{-\alpha})$ as $\lambda \to \infty$. More generally one considers the space $E_{\alpha\rho} = E_{\alpha\rho}(X, \mathscr{E})$ of all $x \in X$ such that

$$\left(\int_1^\infty (\lambda^\alpha E(\lambda, x))^\rho \frac{d\lambda}{\lambda} \right)^{1/\rho} < \infty, \tag{1.1}$$

where $1 \leqslant \rho \leqslant \infty$ (or sometimes $0 < \rho \leqslant \infty$).

In this note we shall consider a variation of this problem. In the definition of $E(\lambda, x)$ we shall only admit y satisfying *additional constraints* $\Gamma_1(y) = 0, ..., \Gamma_K(y) = 0$. Thus we put

$$E_\Gamma(\lambda, x) = \inf\{ \|x - y\|_X : \|y\|_{\mathscr{E}} < \lambda, \Gamma_1(y) = \cdots = \Gamma_K(y) = 0 \}.$$

0021-9045/93 $5.00

Replacing $E(\lambda, x)$ by $E_\Gamma(\lambda, x)$ in (1.1) we then get a sub-space $E_{\Gamma\alpha p}$ of $E_{\alpha p}$. This note deals with the problem of characterizing the space $E_{\Gamma\alpha p}$ in a specific model situation.

We shall let X be the Lebesque space L_p on the real line. As space $\mathscr{E}$ we shall use the space of entire functions of exponential type belonging to L_p. Thus $\mathscr{E}$ consists of all $y \in L_p$ with Fourier transform $y^\wedge$ having compact support. The quasi-norm (or order) of y is given by

$$\|y\|_{\mathscr{E}} = \sup\{|\xi| : \xi \in \operatorname{supp}(y^\wedge)\}.$$

The simplest type of constraints will be of the form $y(s_1) = \cdots = y(s_J) = 0$, where $s_1, ..., s_J$ are given distinct points. For a given j we can also consider constraints of the form $D^{n_1}y(s_j) = \cdots = D^{n_M}y(s_j) = 0$ where $n_1 < n_2 < \cdots < n_M$. The most general type of constraints we will consider are defined as follows. Let $s_1, ..., s_J$ be distinct points. For a given j we put

$$P_{jk}(D) = D^{n_{jk}} - \sum_{r < n_{jk}} a_{jkr} D^r, \qquad k = 1, ..., K_j, \tag{1.2}$$

and

$$\Gamma_{jk}(y) = (P_{jk}(D)\, y)(s_j). \tag{1.3}$$

Here $j = 1, ..., J$ and the numbers n_{jk}, $k = 1, ..., K_j$, are distinct non-negative integers.

Main Result

Let $E(\lambda, x)$ be the best approximation of x in $L_p(\mathbb{R})$ by means of entire functions of exponential type of order λ. Similarly, let $E_\Gamma(\lambda, x)$ be the corresponding best approximation by means of functions satisfying the additional constraints $\Gamma_{jk}(y) = (P_{jk}(D)y)(s_j) = 0$ where $s_1, ..., s_J$ are distinct points and the differential operators P_{jk} are given by (1.2). For a given j the orders n_{jk} of P_{jk} are assumed to be distinct. Then

$$\left(\int_1^\infty (\lambda^\alpha E_\Gamma(\lambda, x))^p \frac{d\lambda}{\lambda}\right)^{1/p} < \infty, \tag{1.4}$$

if and only if the following two conditions hold:

$$\left(\int_1^\infty (\lambda^\alpha E(\lambda, x))^p \frac{d\lambda}{\lambda}\right)^{1/p} < \infty, \tag{1.5}$$

$$\Gamma_{kj}(x)=0, \quad \text{if} \quad \alpha > n_{kj}+1/p,$$
$$\left(\int_0^{\varepsilon}\left(\frac{1}{\tau}\int_{-\tau}^{\tau}|(P_{jk}(D)\,x)(s_j+s)|^p\,ds\right)^{\rho/p}\frac{d\tau}{\tau}\right)^{1/\rho}<\infty, \quad \text{if} \quad \alpha=n_{kj}+1/p, \tag{1.6}$$
$$\text{no condition} \quad \text{if} \quad \alpha<n_{kj}+1/p.$$

In (1.6) we have $j=1, ..., J$, $k=1, ..., K_j$, and $0<\alpha<\infty$, $1\leqslant p\leqslant\infty$, $1\leqslant\rho\leqslant\infty$. The number ε is an arbitrary (small) positive number.

Condition (1.5) can be rephrased in terms of Besov spaces (or interpolation spaces). Let N be any integer such that $N>\alpha$. Then we introduce the modulus of continuity

$$\omega_p^N(t,x)=\sup_{|h|<t}\|\Delta_h^N x\|_p,$$

where Δ_h is the first order difference operator defined by $\Delta_h x(s)=x(s+h)-x(s)$. The Besov space $B_{p,\rho}^{\alpha}$ consists of all $x\in L_p$ such that

$$\left(\int_0^1 (t^{-\alpha}\omega_p^N(t,x))^{\rho}\frac{dt}{t}\right)^{1/\rho}<\infty. \tag{1.7}$$

Conditions (1.5) and (1.7) are equivalent. Using this fact, we get an explicit characterization of the space defined by (1.4).

Similar results were obtained by Grisvard [2] and Löfström [3, 4] in connection with interpolation of boundary value problems.

This work was supported by the Swedish Research Board for Natural Sciences.

2. Two Basic Inequalities

The inequalities of Jackson and Bernstein play an important role in the characterization of the approximation spaces $E_{\alpha\rho}$ (no constraints). To describe these inequalities we introduce generalized Sobolev space H_p^N as follows. Let the operator $|D|$ be defined by $(|D|\,x)^\wedge(\xi)=|\xi|\,x^\wedge(\xi)$. Then H_p^N consists of all L_p-functions x such that $|D|^N x\in L_p$. As semi-norm on H_p^N we use

$$\|x\|_N=\|\,|D|^N x\|,$$

where $\|\ \|$ denotes the norm on L_p. Then the inequalities of Jackson and Bernstein read

$$E(\lambda,x)\leqslant C\lambda^{-N}\|x\|_N, \qquad x\in H_p^N, \tag{2.1}$$
$$\|x\|_N\leqslant C\|x\|_{\mathscr{E}}^N\,\|x\|, \qquad x\in\mathscr{E}. \tag{2.2}$$

These inequalities can be used to characterize the approximation space $E_{\alpha\rho}$ as a Besov space.

We are now going to state and prove the counterparts of the inequalities of Jackson and Bernstein for best approximation with constraints.

Consider the constraint operators Γ_{jk} defined in Section 1, using the differential operators P_{jk} of order n_{jk}. Choose an integer N which is large enough (see below). Then Sobolev's embedding theorem implies that if $x \in H_p^N$ then $P_{jk}x$ are continuous functions on the real line. Thus $\Gamma_{jk}(x)$ are well defined quantities. We now introduce the space $H_{p\Gamma}^N$ of all $x \in H_p^N$ such that $\Gamma_{jk}(x) = 0$ for $j = 1, ..., J$, $k = 1, ..., K_j$. We also consider the space of all $x \in \mathscr{E}$ satisfying these constraints.

THEOREM 1 (The Inequalities of Jackson and Bernstein). *If N is larger than the orders of the differential operators appearing in the definition of the constraints Γ_{jk}, then*

$$E_\Gamma(\lambda, x) \leqslant C\lambda^{-N} \|x\|_N, \qquad \text{if} \quad x \in H_{p\Gamma}^N, \quad \lambda > 1, \tag{2.3}$$

$$\|x\|_N \leqslant C \|x\|_{\mathscr{E}}^N \|X\|, \qquad \text{if} \quad x \in \mathscr{E}_\Gamma, \quad \|x\|_{\mathscr{E}} > 1. \tag{2.4}$$

Proof. Bernstein's inequality (2.4) follows at once from (2.2). For the convenience of the reader we give the simple proof.

Choose a function Φ in the Schwartz class φ such that $\Phi^\wedge(\xi) = 1$ for $|\xi| < 1/2$ and $\Phi^\wedge(\xi) = 0$ for $|\xi| > 1$. Put $\Phi_\lambda(s) = \lambda\Phi(\lambda s)$. Then if $x \in \mathscr{E}_\Gamma$ and $\|x\|_{\mathscr{E}} = \mu$, we have $\Phi_{2\mu} * x = x$. Thus

$$\|x\|_N - \|x\| = \| |D|^N \Phi_{2\mu} * x\| = (2\mu)^N \|(|D|^N \Phi)_{2\mu} * x\|.$$

Since $|D|^N \Phi \in L_1$, we conclude that (2.4) holds.

We now prove Jackson's inequality (2.3). This is easy if we have no constraints, since then we can simply put $y = \Phi_\lambda * x$. Clearly $y \in \mathscr{E}$ and $\|y\|_{\mathscr{E}} < \lambda$. Consequently $E(\lambda, x) \leqslant \|x - y\|$. Now define a function Ψ in φ by writing

$$\Psi^\wedge(\xi) = |\xi|^{-N}(\Phi^\wedge(\xi) - \Phi^\wedge(2\xi)).$$

Then

$$x - y = \sum_{v > 0} (\Phi_{\lambda 2^{v+1}} - \Phi_{\lambda 2^v}) * x = \lambda^{-N} \sum_{v \geqslant 0} 2^{-Nv} \Psi_{\lambda 2^v} * |D|^N x, \tag{2.5}$$

which implies

$$\|x - y\| \leqslant C\lambda^{-N} \|x\|_N. \tag{2.6}$$

This proves (2.1).

To show (2.3) we shall modify y to satisfy the constraints. Let us introduce the functions

$$\varphi_{jk}(\lambda, s) = \Gamma_{jk}(\Phi_\lambda(\cdot - s)), \qquad j = 1, ..., J, \quad k = 1, ..., K_j.$$
$$u_{lm} = \sum_{j,k} c_{lm,jk}\, \bar{\varphi}_{jk}, \qquad l = 1, ..., J, \quad m = 1, ..., K_l.$$

We shall choose the coefficients $c_{lm,jk}$ so that

$$\int_{-\infty}^{\infty} \varphi_{jk}(\lambda, s)\, u_{lm}(\lambda, s)\, ds = \begin{cases} 1 & \text{if } j = l, k = m, \\ 0 & \text{otherwise.} \end{cases} \tag{2.7}$$

We shall prove that this is possible and that

$$\|u_{lm}\| \leqslant C\lambda^{-n_{lm} - 1/p}. \tag{2.8}$$

Accepting this for the moment we put

$$z = y - \sum_{j,k} \Gamma_{jk}(y) \cdot \Phi_\lambda * u_{jk}.$$

Then (2.7) implies that

$$\Gamma_{lm}(z) = \Gamma_{lm}(y) - \sum_{j,k} \Gamma_{j,k}(y) \int_{-\infty}^{\infty} \varphi_{lm}(\lambda, s)\, u_{j,k}(\lambda, s)\, ds = 0,$$

for all l, m. Thus $z \in \mathscr{E}_\Gamma$ and

$$E_\Gamma(\lambda, x) \leqslant \|x - z\| \leqslant \|x - y\| + \sum_{l,m} |\Gamma_{lm}(y)|\, \|\Phi_\lambda * u_{lm}\|.$$

If $x \in H^N_{p\Gamma}$ we have $\Gamma_{lm}(x) = 0$. Thus, using (2.5) we get

$$|\Gamma_{lm}(y)| = |\Gamma_{lm}(x - y)| \leqslant \lambda^{-N} \sum_{v \geqslant 0} 2^{-Nv} \|\psi_{lm}(\lambda 2^v, \cdot)\|'\, \|x\|_N, \tag{2.9}$$

where

$$\psi_{lm}(\mu, s) = \Gamma_{lm}(\Psi_\mu(\cdot - s))$$

and $\| \; \|'$ denotes the norm in $L_{p'}$. Later on we shall prove

$$\|\psi_{lm}(\mu, \cdot)\|' \leqslant C\mu^{n_{lm} + 1/p}. \tag{2.10}$$

Thus (2.9) implies

$$|\Gamma_{lm}(y)| \leqslant C\lambda^{-N + n_{lm} + 1/p}\, \|x\|_N.$$

Using this combined with (2.6) and (2.8), we get (2.3).

It remains to prove (2.7), (2.8), and (2.10). Let us put

$$b_{jk,\,lm} = (\varphi_{jk}, \varphi_{lm}) = \int_{-\infty}^{\infty} \varphi_{jk}(\lambda, s)\, \overline{\varphi_{lm}(\lambda, s)}\, ds,$$

and consider the square matrix B formed by these numbers. For each $\lambda > 0$, B has an inverse, because

$$\begin{aligned}\sum_{jklm} \bar{z}_{jk} b_{jk,\,lm} z_{lm} &= \int_{-\infty}^{\infty} \left| \sum_{jk} z_{jk}\, \varphi_{jk}(\lambda, s)\right|^2 ds \\ &= \int_{-\infty}^{\infty} \left| \sum_{jk} z_{jk}\, P_{jk}(i\xi)\, e^{-is_j\xi}\, \Phi^\wedge(\xi/\lambda)\right|^2 d\xi > 0\end{aligned}$$

for all z such that $\sum |z_{jk}|^2 = 1$. This is the point where we use the fact that the orders n_{jk} of P_{jk} are distinct for a given j.

Now we define the matrix $[c_{jk,\,lm}]$ to be the inverse of B. Writing

$$\Phi_n = D^n \Phi, \qquad n = n_{lm}$$

we have

$$\begin{aligned}\varphi_{lm}(\lambda, s) &= \lambda^n \Big(\Phi_{n,\lambda}(s_l - s) - \sum_{r<n} a_{lmr}\, \lambda^{r-n}\, \Phi_{r,\lambda}(s_l - s)\Big) \\ &= \lambda^{n+1} \Omega_{lm}(\lambda, \lambda s_l - \lambda s).\end{aligned}$$

Thus

$$b_{jk,\,lm} = \lambda^{n_{jk} + n_{lm} + 1} \int_{-\infty}^{\infty} \Omega_{jk}(\lambda, s)\, \overline{\Omega_{lm}(\lambda, \lambda(s_l - s_j) + s)}\, ds.$$

This implies that if $j \neq l$ (i.e., $s_j \neq s_l$), then $b_{jk,\,lm}$ is very small if λ is large. If $j = l$, then $b_{jk,\,lm}$ has the form

$$b_{jk,\,lm} = \lambda^{n_{jk} + n_{lm} + 1} (\beta_{jk,\,lm} + O(\lambda^{-1})).$$

Therefore

$$c_{lm,\,jk} = \lambda^{-n_{jk} - n_{lm} - 1} (\gamma_{lm,\,jk} + O(\lambda^{-1}))$$

and thus

$$u_{lm} = \lambda^{-n_{lm}} \sum_{j,\,k} (\gamma_{l,\,m,\,jk} + O(\lambda^{-1}))\, \overline{\Omega_{jk}(\lambda, \lambda s_j - \lambda s)}.$$

This gives (2.8).

The estimate (2.10) is proved in a similar way since ψ_{lm} has the form

$$\psi_{lm}(\lambda, s) = \lambda^{n_{lm}+1}\tilde{\Omega}_{lm}(\lambda, \lambda s_l - \lambda s). \quad \blacksquare$$

3. Best Approximation and interpolation

Using the inequalities of Jackson and Bernstein, we now describe best approximation spaces in terms of interpolation spaces. Let $E_{\alpha p \Gamma}$ be the space of all $x \in L_p$ such that

$$\left(\int_1^\infty (\lambda^\alpha E_\Gamma(\lambda, x))^p \frac{d\lambda}{\lambda}\right)^{1/p} < \infty, \qquad 1 \leqslant p \leqslant \infty. \tag{3.1}$$

THEOREM 2. *Suppose N is a positive integer and that $0 < \alpha < N$. Then*

$$E_{\alpha p \Gamma} = (L_p, H^N_{p\Gamma})_{\theta, p}, \qquad \theta = \alpha/N. \tag{3.2}$$

Proof. First let us recall the definition of the interpolation space on the right-hand side of (3.2). It is defined by means of the functional

$$K_\Gamma(t, x) = \inf\{\|x - y\| + t\|y\|_N : y \in H^N_{p\Gamma}\}.$$

The space $(L_p, H^N_{p\Gamma})_{\theta, p}$ consists of all $x \in L_p$ such that

$$\left(\int_0^1 (t^{-\theta}K_\Gamma(t, x))^p \frac{dt}{t}\right)^{1/p} < \infty. \tag{3.3}$$

Thus we shall prove that (3.1) and (3.3) are equivalent if $\theta = \alpha/N$.

To prove that (3.3) implies (3.1) we note that for any $y \in H^N_{p\Gamma}$ we have, using Jackson's inequality

$$E_\Gamma(\lambda, x) \leqslant \|x - y\| + E_\Gamma(\lambda, y) \leqslant C(\|x - y\| + \lambda^{-N}\|y\|_N).$$

This implies $E_\Gamma(\lambda, x) \leqslant CK_\Gamma(\lambda^{-N}, x)$.

Conversely assume that $\|x - z\| \leqslant 2E_\Gamma(2^r, x)$ where $z_r \in \mathscr{E}_\Gamma$, $\|z_r\|_{\mathscr{E}} < 2^r$ $(r \geqslant 1)$. Put $z_0 = 0$. Then $x = \sum_{r \geqslant 0}(z_{r+1} - z_r)$. By Bernstein's inequality we have

$$\begin{aligned} K_\Gamma(\lambda^{-N}, z_{r+1} - z_r) &\leqslant \min(\|z_{r+1} - z_r\|, \lambda^{-N}\|z_{r+1} - z_r\|_N) \\ &\leqslant C\min(1, \lambda^{-N}2^{Nr})\|z_{r+1} - z_r\|. \end{aligned}$$

Thus

$$K_\Gamma(\lambda^{-N}, x) \leqslant C\sum_{r \geqslant 0}\min(1, \lambda^{-N}2^{Nr})\,E_\Gamma(2^{r+1}, x).$$

From this estimate we deduce that (3.1) implies (3.3) in the following way. First we observe that

$$\left(\int_0^1 (t^{-\theta} K_\Gamma(t, x))^p \frac{dt}{t}\right)^{1/p}$$
$$\leqslant C\left(\sum_{s\geqslant 0} (2^{\alpha s} K_\Gamma(2^{-Ns}, x))^p\right)^{1/p}$$
$$\leqslant C\left(\sum_{s\geqslant 0}\left(\sum_{r\geqslant 0} 2^{\alpha(s-r)} \min(1, 2^{N(r-s)})\, 2^{\alpha r} E_\Gamma(2^r, x)\right)^p\right)^{1/p}.$$

The right-hand side can be viewed as the norm of a convolution between the sequences $2^{-\alpha r}\min(1, 2^{Nr})$ and $2^{\alpha r} E_\Gamma(2^r, x)$. Since

$$\sum_{-\infty}^{\infty} 2^{-\alpha r} \min(1, 2^{Nr}) < \infty$$

we conclude that

$$\left(\int_0^1 t^{-\theta} K_\Gamma(t, x))^p \frac{dt}{t}\right)^{1/p} \leqslant C\left(\sum_{r\geqslant 0} (2^{\alpha r} E_\Gamma(2^r, x))^p\right)^{1/p}.$$

This proves the implication from (3.1) to (3.3).

4. Reduction to a Single Point

We consider a set Γ of constraints of a particular form. They are given by means of distinct points s_j, $j = 1, ..., J$, and differential operators P_{jk} by means of the formula $\Gamma_{jk}(x) = (P_{jk}(D)\,x)\,(s_j) = 0$. For a given m ($m = 1, ..., J$) we let Γ_j be the set of all constraints Γ_{jk}, $k = 1, ..., K_j$, and $H^N_{p\Gamma_j}$ be the space of all $x \in H^N_p$ such that $\Gamma_{jk}(x) = 0$ for $k = 1, ..., K_j$. We claim that

$$(L_p, H^N_{p\Gamma})_{\theta, p} = \bigcap_{m=1}^{M} (L_p, H^N_{p\Gamma_m})_{\theta, p}. \tag{4.1}$$

This result makes it possible to reduce the proof of our main result to a single point s_j.

To prove (4.1) let x belong to the intersection on the right-hand side. Assume that $y_j \in H^N_{p\Gamma_j}$ and

$$\|x - y_j\| + t\|y_j\|_N \leqslant 2K_{\Gamma_j}(t, x), \qquad j = 1, ..., J.$$

We also assume that $y_0 \in H_p^N$ and $\|x-y_0\| + t\|y_0\|_N \leqslant 2K(t, x)$ where $K(t, x) = \inf(\|x-y\| + t\|y\|_N)$ (no constraints). Let χ_j be an infinitely differentiable function with compact supports in a small neighbourhood of s_j and assume that χ_j is identically 1 in a neighbourhood of s_j. Put $\chi_0 = 1 - \sum \chi_j$ and $z = \chi_0 y_0 + \chi_1 y_1 + \cdots + \chi_J y_J$. Then $\Gamma_j(z) = 0$ for all j and thus

$$K_\Gamma(t, x) \leqslant \|x-z\| + t\|z\|_N \leqslant \sum_{k=0}^{M} (\|\chi_k(x-y_k)\| + \|\chi_k y_k\|_N)$$

$$\leqslant C \sum_{k=0}^{M} (\|x-y_k\| + t\|y_k\|_N).$$

This proves that $x \in (L_p, H_{p\Gamma}^N)_{\theta,p}$. The converse inclusion is trivial.

In order to prove our main result it is now enough to consider a single constraint Γ of the form $\Gamma_k(x) = (P_k(D)\,x)(0) = 0$ where

$$P_k(D)\,x = D^{n_k} - \sum_{r<n_k} a_r D^r x, \qquad 0 \leqslant n_1 < \cdots < n_M < N.$$

The rest of the paper will be devoted to the proof of the following theorem.

THEOREM 3. *The interpolation space* $(L_p, H_{p\Gamma}^N)_{\theta,p}$ *such that*

$$\Gamma_k(x) = (P_k(D)\,x)(0) = 0, \qquad \text{if} \quad \theta N > n_{k_j} + 1/p,$$

$$\left(\int_0^\varepsilon \left(\frac{1}{\tau}\int_{-\tau}^{\tau} |P_k(D)\,x(s)|^p\,ds\right)^{\rho/p} < \infty, \qquad \text{if} \quad \theta N = n_k + 1/p,$$

$$\textit{no condition} \qquad \text{if} \quad \theta N < n_k + 1/p.$$

Clearly this result, in combination with (4.1) will give our main result.

5. PROOF OF THEOREM 3

As a first step towards the proof of Theorem 3 in the previous section, we shall characterize the interpolation spaces $(L_p, H_p^N)_{\theta,p}$ and $(L_p, H_{p\Gamma}^N)_{\theta,p}$ in a way that makes it easy to compare these two spaces.

Let us put $H_{p0}^N = H_p^N$ and define H_{pm}^N as the space of all $x \in H_p^N$ such that $\Gamma_k(x) = 0$ for $k = 1, \ldots, m$. Here $\Gamma_k(x) = (P_k(D)\,x)(0)$ and P_k are differential operators of order n_k $(0 \leqslant n_1 < \cdots < n_M < N)$. Then we shall recursively characterize the spaces $(L_p, H_{pm}^N)_{\theta,p}$, $m = 0, 1, \ldots, M$, by means of a family $\Lambda_m(t)$ of operators and a corresponding functional L_m defined by

$$L_m(t, x) = \|x - \Lambda_m(t)\,x\| + t\|\Lambda_m(t)\,x\|_N, \qquad 0 < t < \infty. \tag{5.1}$$

LEMMA 1. *Suppose $A_m(t)$ is strongly continuous on L_p and that $A_m(t)$ maps L_p into H^N_{pm} for each $t>0$. Moreover, assume that*

$$\begin{aligned} &\|x - A_m(t)\,x\|,\ t\|A_m(t)\,x\|_N \leqslant C\|x\| && \text{if}\quad x \in L_p, \\ &\|x - A_m(t)\,x\|,\ t\|A_m(t)\,x\|_N \leqslant Ct\|x\|_N && \text{if}\quad x \in H^N_{pm}. \end{aligned} \tag{5.2}$$

Then $x \in (L_p, H^N_{pm})_{\theta,p}$ if and only if $x \in L_p$ and

$$\left(\int_0^1 \left(t^{-\theta} L_m(t, x)^p \frac{dt}{t}\right)\right)^{1/p} < \infty.$$

Proof. See [1]. ∎

In order to define the operator $A_0(t)$ we introduce the function H^τ by the formulas

$$H^\wedge(\xi) = (1 + \xi^{2N})^{-1}, \qquad H^\tau(s) = \tau^{-1} H(s/\tau). \tag{5.3}$$

Then we put $A_0(t)\,x = H^\tau * x$, if $t = \tau^N$. Then it is easy to see that (5.2) holds with $m = 0$. Next we define $A_m(t)$ by a construction which is similar to the one used in the proof of Theorem 2. We put

$$\varphi_\nu(\tau, \sigma) = \overline{(P_\nu(D)\, H^\tau)(-\sigma)} \tag{5.4}$$

$$A_m(t)\,x = H^\tau * x - \sum_{k=1}^{m} (x, \varphi_k) \cdot H^\tau * u_k, \tag{5.5}$$

where the functions u_k are defined by

$$u_k = \sum_{\nu=1}^{m} c_{k\nu}\, \varphi_\nu, \tag{5.6}$$

$$(u_k, \varphi_\nu) = \delta_{k\nu}. \tag{5.7}$$

This means that the matrix $[c_{k\nu}]$ is the inverse of the matrix $[b_{\nu k}]$ where $b_{\nu k} = (\varphi_\nu, \varphi_k)$. The existence of an inverse is proved in the same way as in the proof of Theorem 2. It follows that

$$\|u_k\| \leqslant c\tau^{n_k + 1/p}. \tag{5.8}$$

As a consequence we get

$$\|x - A_m(t)\,x\| \leqslant \|x - H^\tau * x\| + c \sum_{k=1}^{m} \tau^{n_k + 1/p}\, |(x, \varphi_k)|, \qquad t = \tau^N,$$

and

$$t\|A_m(t)\,x\|_N \leqslant t\|H^\tau * x\|_N + c\sum_{k=1}^{m} \tau^{n_k+1/p}\,|(x, \varphi_k)|.$$

Moreover we have $(P_v(D)\,A_m(t)\,x)(0)=0$. In order to show that $A_m(t)$ satisfies the assumptions of Lemma 1 it is enough to show that

$$|(x, \varphi_k)| \leqslant C\tau^{-n_k-1/p}\min(\|x\|, \tau^N\|x\|_N).$$

This is easily seen in the following way. First note that

$$|(x, \varphi_k)| \leqslant \|x\|\;\|\varphi_k\|' \leqslant C\tau^{-n_k-1/p}\,\|x\|.$$

If $x\in H^N_{pm}$ we have $(P_k(D)\,x)(0)=0$. Now note that $y * H^\tau = y-\tau^{2N}\,|D|^N\,y * |D|^N\,H^\tau$. Thus we have

$$\begin{aligned}(x, \varphi_k) &= (P_k(D)\,x * H^\tau)(0)\\ &= (P_k(D)\,x)(0)-\tau^{2N}(|D|^N\,P_k(D)\,x * |D|^N\,H^\tau)(0)\\ &= -\tau^{2N}(|D|^N\,x * P_k(D)\,|D|^N\,H^\tau)(0).\end{aligned}$$

Consequently,

$$|(x, \varphi_k)| \leqslant C\tau^{N-n_k-1/p}\,\|x\|_N \qquad \text{if} \quad x\in H^N_{pm}.$$

We have now proved that $A_m(t)$ satisfies the assumptions of Lemma 1. As a consequence we get the following lemma.

LEMMA 2. *Let H^τ be defined by* (5.3). *Then $x\in(L_p, H^N_{pm})_{\theta,\rho}$ if and only if*

$$x\in(L_p, H^N_p)_{\theta,\rho}\left(\int_0^1\left(\tau^{-N\theta+n_k+1/p}\left|\int_{-\infty}^{\infty} H^\tau(-\sigma)\,P_k(D)\,x(\sigma)\,d\sigma\right|\right)^\rho d\tau/\tau\right)^{1/\rho}$$
$$<\infty \qquad \textit{for} \quad k=1, ..., m.$$

Proof. First note that $(L_p, H^N_{pm})_{\theta,\rho}\subset(L_p, H^N_{pm-1})_{\theta,\rho}$. Let L_m be defined by (5.1). By inspection of the difference between L_m and L_{m-1} one easily sees that $x\in(L_p, H^N_{pm})_{\theta,\rho}$ if and only if $x\in(L_p, H_{pm-1})_{\theta,\rho}$ and

$$\left(\int_0^1 (\tau^{-N\theta}|(x, \varphi_m)|\cdot\|H^\tau * u_m\|)^\rho\,d\tau/\tau\right)^{1/\rho}<\infty,$$

where $\|H_\tau * u_m\| \leqslant C\tau^{n_m+1/p}$. Since $(x, \varphi_m)=(P_m(D)\,x * H^\tau)(0)$, we get the conclusion of the lemma. ∎

The proof of Theorem 3 is now nearly complete. We have only to apply the following result.

LEMMA 3. *Assume that* $y \in (L_p, H_p^N)_{\eta, \rho}$. *Then*

$$R_{\eta, \rho}(y) = \left(\int_0^1 \left(\tau^{-N\eta + 1/p} \left| \int_{-\infty}^{\infty} H^{\tau}(-\sigma)\, y(\sigma)\, d\sigma \right|\right)^{\rho} d\tau/\tau\right)^{1/\rho} < \infty$$

if and only if

$$\begin{aligned} & y(0) = 0 && \textit{if } N\eta > 1/p, \\ & \left(\int_0^{\varepsilon} \left(\frac{1}{\tau}\int_{-\tau}^{\tau} |y(\sigma)|^p\, d\sigma\right)^{\rho/p} d\tau/\tau\right)^{1/\rho} < \infty, && \textit{if } N\eta = 1/p, \\ & \textit{no condition} && \textit{if } N\eta < 1/p. \end{aligned}$$

Proof. First consider the case $\alpha = N\eta - 1/p > 0$. Suppose $y \in (L_p, H_p^N)_{\eta, \rho}$ and $R_{\eta, \rho}(y) < \infty$. Then y is a continuous function and

$$\lim_{\tau \to 0} \int_{-\infty}^{\infty} H^{\tau}(-\sigma)\, y(\sigma)\, d\sigma = c y(0), \qquad c \neq 0.$$

Consequently $R_{\eta, \rho}(y) < \infty$ implies $y(0) = 0$ if $\alpha > 0$. Conversely assume $y(0) = 0$. Then we put $z(s) = y(s) - y(2^{-1}s)$ and $z_k(s) = z(2^{-k}s)$. Then $y = \sum_{k \geqslant 0} z_k$ and $z \in (H_p, H_{p\Gamma}^N)_{\eta, \rho}$ where $\Gamma(z) = z(0) = 0$. Then Lemma 2 implies that $R_{\eta, \rho}(z) < \infty$. Thus

$$R_{\eta, \rho}(y) \leqslant \sum_{k \geqslant 0} R_{\eta, \rho}(z_k) = \sum_{k \geqslant 0} 2^{-\alpha k} R_{\eta, \rho}(z) < \infty.$$

Next consider the case $\alpha = N\eta - 1/p < 0$. Then put $z(s) = y(s) - y(2s)$ and $z_k(s) = z(2^k s)$. Then $y = \sum_{k \geqslant 0} z_k$ in L_p and $z \in (H_p, H_{p\Gamma}^N)_{\eta, \rho}$ so that $R_{\eta, \rho}(z) < \infty$. Thus

$$R_{\eta, \rho}(y) = \sum_{k \geqslant 0} R_{\eta, \rho}(z_k) = \sum_{k \geqslant 0} 2^{\alpha k} R_{\eta, \rho}(z) < \infty.$$

Finally we consider the case $\alpha = 0$. Then partial integration gives

$$\begin{aligned} \left| \int_0^{\infty} H^{\tau}(-\sigma)\, y(\sigma)\, d\sigma \right| &\leqslant \left| \int_0^{\infty} DH(\sigma) \frac{1}{\tau} \int_0^{\sigma\tau} y(s)\, ds\, d\sigma \right| \\ &\leqslant \int_0^{\infty} \sigma |DH(\sigma)| \left(\frac{1}{\sigma\tau} \int_0^{\sigma\tau} |y(s)|^p\, ds\right)^{1/p} d\sigma. \end{aligned}$$

It follows that

$$\left|\int_{-\infty}^{\infty} H^{\tau}(-\sigma)\, y(\sigma)\, d\sigma\right| \leqslant \int_{-\infty}^{\infty} |\sigma|\; |DH(\sigma)| \left(\frac{1}{\sigma\tau}\int_{-\sigma\tau}^{\sigma\tau} |y(s)|^{p}\, ds\right)^{1/p} d\sigma,$$

which implies $R_{\eta,\rho}(y) < \infty$.

Still considering the case $\alpha = 0$, i.e., $N_\eta = 1/p$, we shall now prove that $R_{\eta,\rho}(y) < \infty$ implies that

$$\left(\int_0^{\varepsilon}\left(\frac{1}{\tau}\,|y(s)|^{p}\, ds\right)^{\rho/p}\frac{d\tau}{\tau}\right)^{1/\rho} < \infty.$$

If $R_{\eta,\rho}(y) < \infty$ we have $y \in (L_p, H^N_{p\Gamma})_{\eta,\rho}$ where $\Gamma y = y(0)$ (by Lemma 2). It is a routine matter to prove that $y \in (L_p, H^1_{p\Gamma})_{1/p,\rho}$. For the convenience of the reader we provide the details here. Using the so called J-method of interpolation theory (see [1, Sect. 3.2]) any $y \in (L_p, H^N_{p\Gamma})_{\eta,\rho}$ can be represented as

$$y(s) = \int_0^{\infty} u(\tau, s)\, d\tau/\tau,$$

where $u(\tau, 0) = 0$ and

$$\left(\int_0^{\infty} (\tau^{-1/p} \max(\|u(\tau)\|, \tau^{N}\|u(\tau)\|_N))^{\rho}\,\frac{d\tau}{\tau}\right)^{1/\rho} < \infty.$$

Using Kolmogorof's inequality we have

$$\max(\|u(\tau)\|, \tau\|u(\tau)\|_1) \leqslant C \max(\|u(\tau)\|, \tau^{N}\|u(\tau)\|_N).$$

Thus

$$\left(\int_0^{\infty} (\tau^{-1/p} \max(\|u(\tau)\|, \tau\|u(\tau)\|_1))^{\rho}\,\frac{d\tau}{\tau}\right)^{1/\rho} < \infty$$

which implies $y \in (L_p, H^1_{p\Gamma})_{1/p,\rho}$. By definition this means that $y = y_0 + y_1$ where $y_1 \in H^1_p$, $y(0) = 0$. Given any function z we define $\tilde{z}$ by

$$\tilde{z}(s) = \begin{cases} z(s) & \text{if } s \geqslant 0, \\ 0 & \text{if } s < 0. \end{cases}$$

Then $\tilde{y}_1 \in H^1_p$, $\tilde{y}_0 \in L_p$, in both cases with equal or smaller norms. Thus $\tilde{y} = \tilde{y}_0 + \tilde{y}_1 \in (L_p, H^1_p)_{1/p,\rho}$. But this implies that

$$\left(\int_{-\infty}^{\infty}\left(\tau^{-1/p}\left(\int_{-\infty}^{\infty} |\tilde{y}(s+\tau) - \tilde{y}(s)|^{p}\, ds\right)^{1/p}\right)^{\rho}\frac{d\tau}{\tau}\right)^{1/\rho} < \infty,$$

which gives

$$\left(\int_0^\infty \left(\frac{1}{\tau}\int_0^\tau |y(s)|^p\, ds\right)^{\rho/p} \frac{d\tau}{\tau}\right)^{1/\rho} < \infty.$$

Replacing $y(s)$ by $y(-s)$ we get

$$\left(\int_0^\infty \left(\frac{1}{\tau}\int_{-\tau}^\tau |y(s)|^p\, ds\right)^{\rho/p} \frac{d\tau}{\tau}\right)^{1/\rho} < \infty. \blacksquare$$

We can now conclude the proof of Theorem 3, by applying Lemma 3 with $y = P_k(D)\,x$ and $\eta = \theta - \eta_k/N$.

References

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